



Stochastic sensitivity analysis of noise-induced intermittency and transition to chaos in one-dimensional discrete-time systems



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ABSTRACT

We study a phenomenon of noise-induced intermittency for the stochastically forced one-dimensional discrete-time system near tangent bifurcation. In a subcritical zone, where the deterministic system has a single stable equilibrium, even small noises generate large-amplitude chaotic oscillations and intermittency. We show that this phenomenon can be explained by a high stochastic sensitivity of this equilibrium. For the analysis of this system, we suggest a constructive method based on stochastic sensitivity functions and confidence intervals technique. An explicit formula for the value of the noise intensity threshold corresponding to the onset of noise-induced intermittency is found. On the basis of our approach, a parametrical diagram of different stochastic regimes of intermittency and asymptotics are given.

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1. Introduction

Due to the interaction between nonlinearity and stochasticity, noise can induce a number of interesting unexpected phenomena in dynamical systems, such as noise-induced transitions [1,2], noise-induced resonance [3–5], noise-induced excitement [6], noise-induced order [7,8] and chaos [9,10]. The transition to chaos is a fundamental and widely studied problem in deterministic nonlinear dynamics. Among the possible routes to chaos is an intermittency route. The system demonstrating intermittent behavior remains for a long duration in some regular regime (laminar state) and at unpredictable moments begins to exhibit chaotic oscillations (turbulent state) before returning to the laminar state. Pomeau and Manneville [11,12] have proposed a simple deterministic one-dimensional model and classified three different types of intermittency. These types (I, II and III) correspond to a tangent bifurcation, a subcritical Hopf bifurcation, or an inverse period-doubling bifurcation. A renormalization group approach to analyze type-I intermittency has been used in Refs. [13,14].

In this paper, we focus on the study of the noise-induced type-I intermittency phenomenon. An influence of noise on the intermittent behavior of nonlinear dynamical systems has been widely studied [15–21].

Frequently, noise-induced intermittency is caused by the multistability of the initial nonlinear deterministic system. Indeed, let the system have coexisting regular (equilibrium or limit cycle) and chaotic attractors. Due to random disturbances, a phase trajectory can cross a separatrix between basins of the attraction and exhibit a new dynamical regime which has no analog in the deterministic case. Random trajectories hopping between basins of coexisting deterministic attractors form a new stochastic attractor. This stochastic attractor joins together two types of dynamics. Trajectories in this attractor exhibit the alternation of phases of noisy regular and noisy chaotic dynamics near initial deterministic attractors and define corresponding type of noise-induced intermittency.

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However, the multistability is not an obligatory condition of the noise-induced intermittency. The phenomenon of noise-induced intermittency can be observed in the specific dynamical systems with a single stable equilibrium only. For these systems, a basin of attraction of equilibrium can be separated on two zones. If the initial point belongs to the first zone localized near the equilibrium, the system quickly relaxes back into the stable equilibrium. Once the initial point lies in second zone, a large excursion of the trajectory is observed. In this case, the system demonstrates high-amplitude oscillations until the trajectory returns to the first zone. Under the small random disturbances, trajectory of this type system leaves a stable equilibrium and forms some probabilistic distribution around it. This noisy equilibrium is localized in the first zone. Once the noise intensity exceeds a certain threshold, the random trajectory hits at second zone and exhibits long-time noisy oscillations until return to first zone and so on. In such a way the stochastically forced system with super-threshold noise demonstrates noise-induced intermittency. Under the random disturbances, this system is transformed from order to chaos. The standard model with this type noise-induced intermittency is a one-dimensional map in a zone of tangent bifurcation. Similar phenomena when small noises generate large-amplitude oscillations can also be observed in continuous-time systems with a single stable equilibrium. The FitzHugh–Nagumo model is a well known example of such noise-induced excitement [22,6].

A probabilistic analysis of the noise-induced phenomena is based on the investigation of corresponding stochastic attractors. A detailed description of stochastic attractors for continuous-time systems is given by the Kolmogorov–Fokker–Planck equation. For discrete-time systems, this description is given by the corresponding integral equation with Frobenius–Perron operator. However, a direct usage of these equations is very difficult even for the simplest cases. To avoid this complexity, various asymptotics and approximations can be considered [23,24].

A stochastic sensitivity function (SSF) method has been used for the constructive probabilistic description of stochastic attractors for both continuous [25] and discrete-time [26] systems. The aim of our work is to demonstrate how the SSF technique can be applied to the parametrical analysis of the noise-induced intermittency for discrete-time nonlinear systems. Our general approach is illustrated on the example of the simple one-dimensional model.

In Section 2, we introduce this model and discuss phenomena of noise-induced intermittency and noise-induced chaotization in a subcritical zone near the tangent bifurcation.

The main results of our paper are shown in Section 3.

In Section 3.1, we present a brief theoretical background of the general SSF technique for stochastic equilibria of discrete-time dynamical systems. A constructive description of the dispersion of random states in the stochastic equilibria is given by confidence intervals. The size of the confidence interval is defined by the noise intensity, value of stochastic sensitivity and fiducial probability.

In Section 3.2, this technique is applied to the detailed parametrical analysis of noise-induced intermittency for the one-dimensional model introduced in Section 2. Through this study, we find an explicit formula for the value of noise intensity threshold corresponding the onset of noise-induced intermittency and construct a parametrical diagram of different stochastic regimes.

In Section 3.3, constructive abilities of our approach for the asymptotic analysis of the noise-induced intermittency in a tangent bifurcation zone for the general one-dimensional systems are demonstrated.

2. Phenomena of noise-induced intermittency and chaotization

2.1. Deterministic model. Intermittency

We consider a discrete-time nonlinear dynamic system

$$x_{t+1} = f(x_t, \mu), \quad f(x, \mu) = \mu x(1-x)(lx^2 + px + q), \quad (1)$$

where

$$l = \frac{1}{1-s_1+s_2-s_3}, \quad p = l(1-s_1), \quad q = l(1-s_1+s_2), \\ s_1 = \bar{x}_1 + \bar{x}_2 + \bar{x}_3, \quad s_2 = \bar{x}_1\bar{x}_2 + \bar{x}_2\bar{x}_3 + \bar{x}_3\bar{x}_1, \quad s_3 = \bar{x}_1\bar{x}_2\bar{x}_3.$$

For any μ , the system (1) has a trivial equilibrium $\bar{x}_0 = 0$. Values $\bar{x}_1, \bar{x}_2, \bar{x}_3$ ($\bar{x}_1 \leq \bar{x}_2 \leq \bar{x}_3$) are nontrivial equilibria of the system (1) for $\mu = 1$. As parameter μ varies near $\mu = 1$, these equilibria change too. So we denote the corresponding functions by $\bar{x}_1(\mu), \bar{x}_2(\mu), \bar{x}_3(\mu)$.

This system is a convenient model for the study of the phenomenon of intermittency.

We fix values $\bar{x}_1 = \bar{x}_2 = 0.25$, $\bar{x}_3 = 0.85$ and vary the parameter μ near the value $\mu_* = 1$. Note that for the interval $\mu_* - 0.1 < \mu < \mu_* + 0.1$, the equilibria \bar{x}_0 and $\bar{x}_3(\mu)$ are unstable.

Here, the value $\mu_* = 1$ is a tangent bifurcation point (see Fig. 1). For $\mu < \mu_*$, we have $\bar{x}_1(\mu) < \bar{x}_2(\mu)$, where the equilibrium $\bar{x}_1(\mu)$ is stable (black circle) and equilibrium $\bar{x}_2(\mu)$ is unstable (white circle). For $\mu = \mu_*$, these equilibria coalesce into the single semistable equilibrium $\bar{x}_1(\mu_*) = \bar{x}_2(\mu_*)$. For $\mu > \mu_*$, this equilibrium disappears.

In Fig. 2, the attractors of the system (1) for $\mu \in (0.995, 1.005)$ are presented.

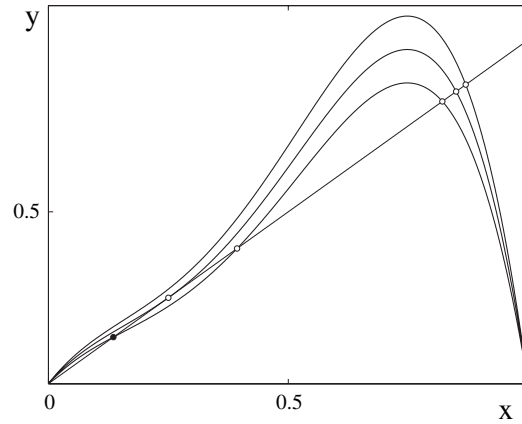


Fig. 1. Plots of $y = f(x, \mu)$ for $\mu > \mu_*$ (upper); $\mu = \mu_*$ (middle); $\mu < \mu_*$ (lower).

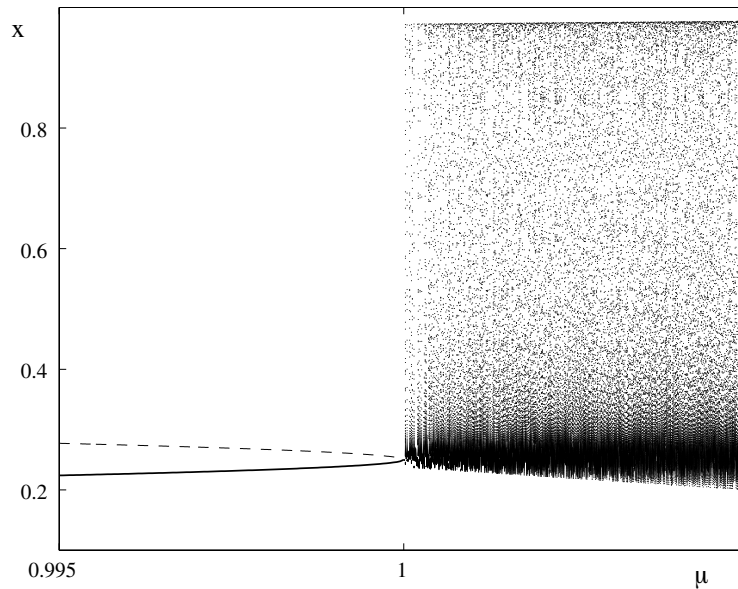


Fig. 2. Attractors of deterministic system (1).

At the left subinterval $0.995 \leq \mu \leq 1$, the system (1) has a stable equilibrium $\bar{x}_1(\mu)$ (lower solid curve) and unstable equilibrium $\bar{x}_2(\mu)$ (dashed curve). In Fig. 3, iterations of the system (1) with $\mu = 0.996$ are shown. Here unstable equilibria \bar{x}_2, \bar{x}_3 are plotted by white circles and a stable equilibrium \bar{x}_1 is plotted by the black circle. There are two different roads towards the stable equilibrium \bar{x}_1 depending on the initial state x_0 location. For $x_0 \in (0, \bar{x}_2)$, one can observe a short monotonous walk. An example of such walk for the initial state $x_0 = 0.04$ is shown in Fig. 3. The interval $x_0 \in (0, \bar{x}_2)$ is a basin of the monotonous attraction for the stable equilibrium \bar{x}_1 . But for $x_0 \in (\bar{x}_2, \bar{x}_3)$ we have a long excursion away from the equilibrium \bar{x}_1 with oscillations until the trajectory comes after all to the interval $(0, \bar{x}_2)$ (see a trajectory with an initial state $x_0 = 0.4$ in Fig. 3). The unstable equilibrium \bar{x}_2 separates these two types of trajectories.

As the parameter μ increases, the critical value $\mu_* = 1$ is reached. At this critical value the stable equilibrium $\bar{x}_1(\mu)$ coalesces with the unstable equilibrium $\bar{x}_2(\mu)$. Above the critical value $\mu_* = 1$, these merged equilibria disappear and a new chaotic attractor is born. A transition of the parameter μ across the value $\mu_* = 1$ produces the sudden creation of the strange attractor with a wide size. For μ slightly larger than $\mu_* = 1$, the orbit of this new attractor typically spends the long stretched time in the vicinity of the point $\bar{x}_1(\mu_*)$ with the slow monotonous increase. At the end of this laminar time interval, the orbit suddenly bursts out of this region and switches to chaotic oscillations around the unstable equilibrium $\bar{x}_3(\mu)$. At the end of this turbulent interval, the orbit hits the vicinity of point $\bar{x}_1(\mu_*)$ again and so on. In Fig. 4, the dynamics of the system (1) with $\mu = 1.005$ is shown. Here one can see iterations and time series for the initial state $x_0 = 0.04$ (Fig. 4(a), (b)). A distribution of states in the chaotic attractor is plotted in Fig. 4(c). This sporadic switching between two qualitatively different behaviors of the system (1) is a typical example of Pomeau and Manneville type I intermittency under a tangent bifurcation.

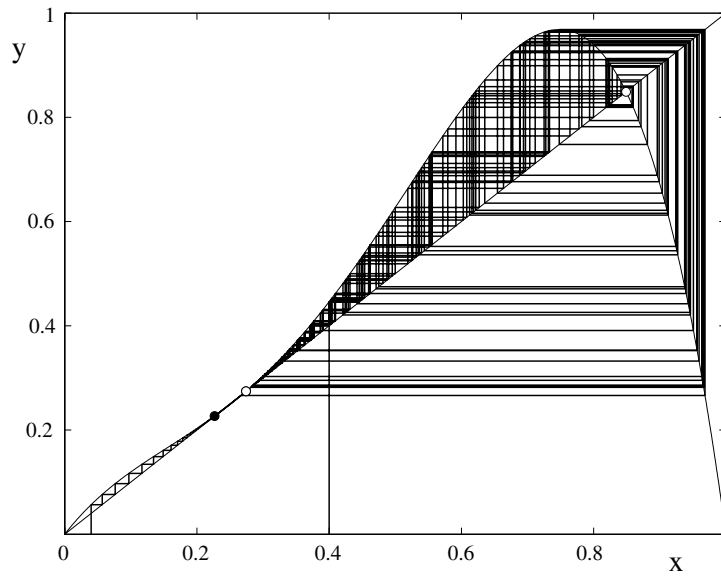


Fig. 3. Iterations of deterministic system (1) with $\mu = 0.996$.

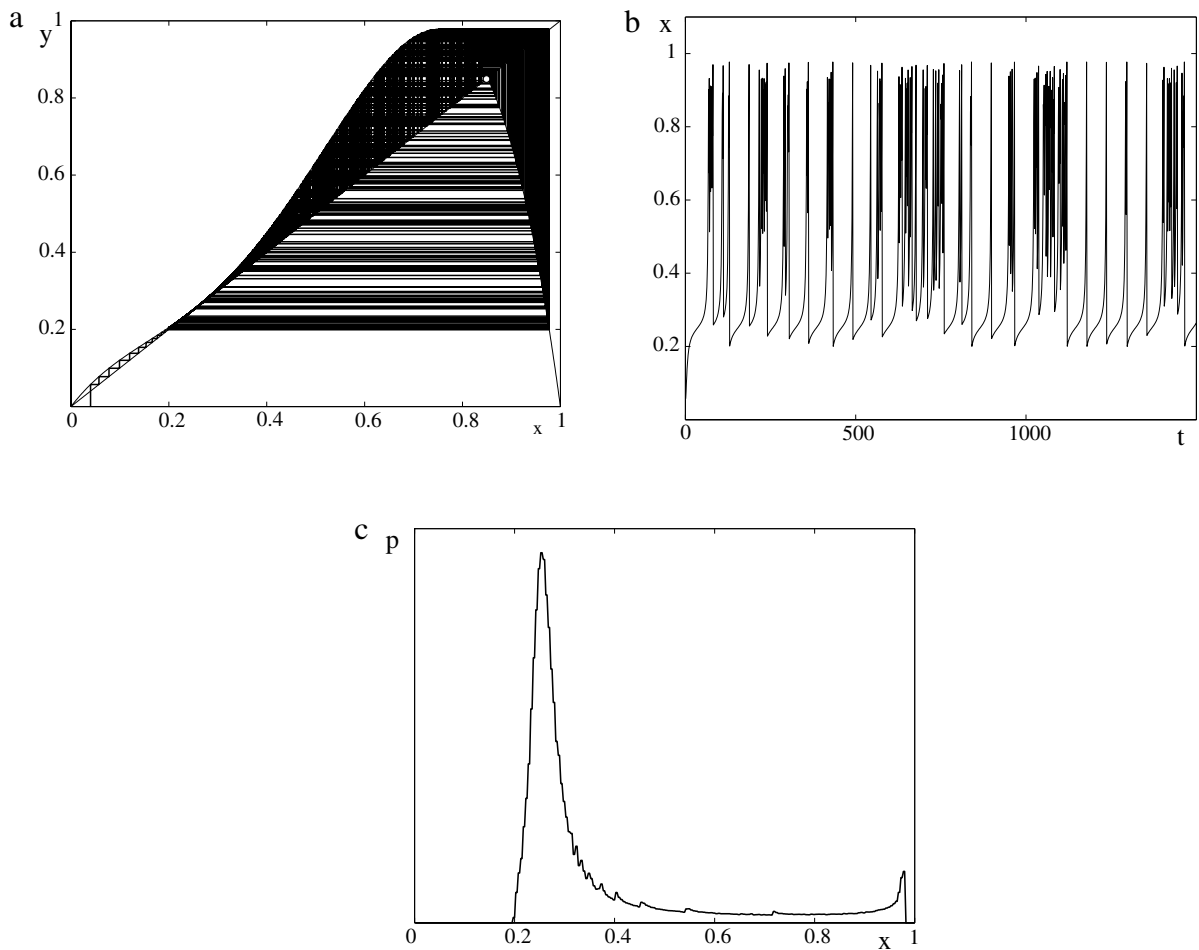


Fig. 4. Dynamics of system (1) with $\mu = 1.005$: (a) iterations; (b) time series; (c) distribution of states in attractor.

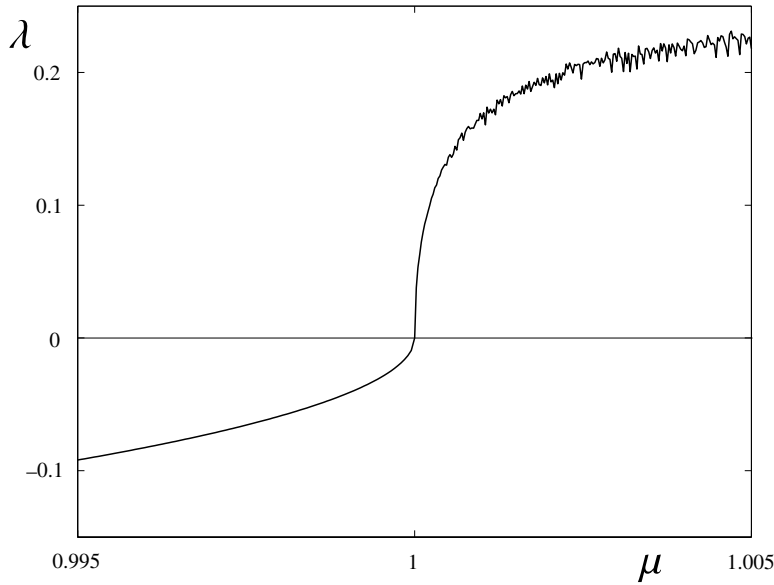


Fig. 5. Lyapunov exponent of deterministic system (1).

In Fig. 5, a plot of the Lyapunov exponent for attractors of the system (1) is presented. A positiveness of this exponent for $\mu > \mu_* = 1$ confirms the chaotic character of attractors in this parametrical zone.

2.2. Stochastic model. Noise-induced intermittency and transition to chaos

Along with the deterministic system (1) we consider a corresponding stochastic system forced by additive noise

$$x_{t+1} = f(x_t, \mu) + \varepsilon \xi_t, \quad (2)$$

where ξ_t is uncorrelated Gaussian random process with parameters $E\xi_t = 0$, $E\xi_t^2 = 1$, ε is a scalar parameter of the noise intensity.

We study a behavior of this stochastic model for different sets of parameters μ and ε . In Figs. 6 and 7, stochastic attractors and Lyapunov exponents of the system (2) are plotted on the interval $0.995 \leq \mu \leq 1.005$ for three values of the noise intensity $\varepsilon = 0.001, 0.002, 0.005$.

As we can see, noise deforms the deterministic attractor (compare Figs. 2 and 6). As noise intensity increases, a border between order and chaos moves to the left. The changes of the arrangement of attractors are accompanied by the changes in dynamical characteristics (compare Lyapunov exponents in Figs. 5 and 7).

The most essential difference between stochastic and deterministic attractors is observed near the bifurcation point $\mu_* = 1$. The underlying reason is that in the vicinity of this bifurcation point attractors are highly sensitive to random disturbances. Consider in detail a behavior of stochastic system (2) near $\mu_* = 1$. Compare the stochastic response of this system for two fixed values $\mu = 0.998$ and $\mu = 0.9998$. In Figs. 8 and 9, time series are presented for different values of noise intensity.

Consider $\mu = 0.998$. For low noise $\varepsilon = 0.002$, random states are concentrated near the stable deterministic equilibrium \bar{x}_1 (see Fig. 8(a)). For $\varepsilon = 0.005$ one can see stochastic oscillations of large amplitude. Indeed, as the noise intensity increases, the dispersion of random states near \bar{x}_1 grows and iterations with high probability jump over the unstable equilibrium \bar{x}_2 and continue a long excursion with high amplitude oscillations around \bar{x}_3 . After these oscillations, iterations come to the vicinity of the point \bar{x}_1 again and so on (see Fig. 8(b)). In this case, the stochastic model (2) exhibits a coexistence of two different dynamical regimes even if the deterministic system (1) has a stable equilibrium only. This type of dynamics of the system (2) can be determined as a *noise-induced intermittency*.

In Fig. 9, time series of the stochastic system (2) with $\mu = 0.9998$ for $\varepsilon = 0.0005$ and $\varepsilon = 0.002$ are plotted. As can be seen, noise-induced intermittency for this $\mu = 0.9998$ is observed for the lower noise intensity.

For stochastic attractors and their dynamic characteristics, a dependence on noise level is illustrated in Figs. 10 and 11 for $\mu = 0.998$ and $\mu = 0.9998$. In Fig. 10, one can see a sharp growth of the size of the attractor as noise intensity exceeds some critical value. A change of the sign of Lyapunov exponent from minus to plus can be interpreted as a transition from regular to noise-induced chaotic regime (see Fig. 11).

Thus, the results presented here give us a qualitative description of noise-induced transitions from the regular regime to intermittency. More detailed quantitative analysis of noise-induced intermittency will be presented in Section 3 with the help of the SSF technique.

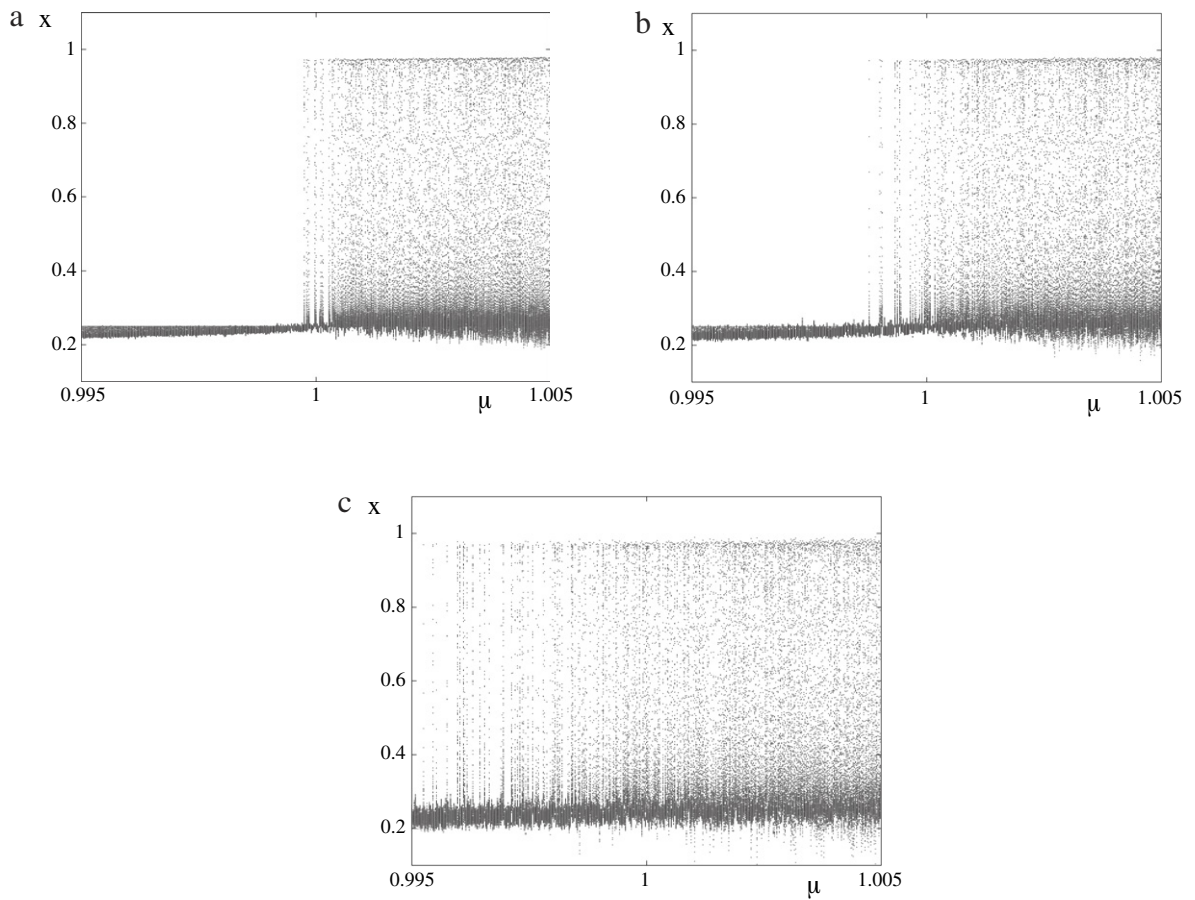


Fig. 6. Attractors of stochastic system (2) for (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.002$, (c) $\varepsilon = 0.005$.

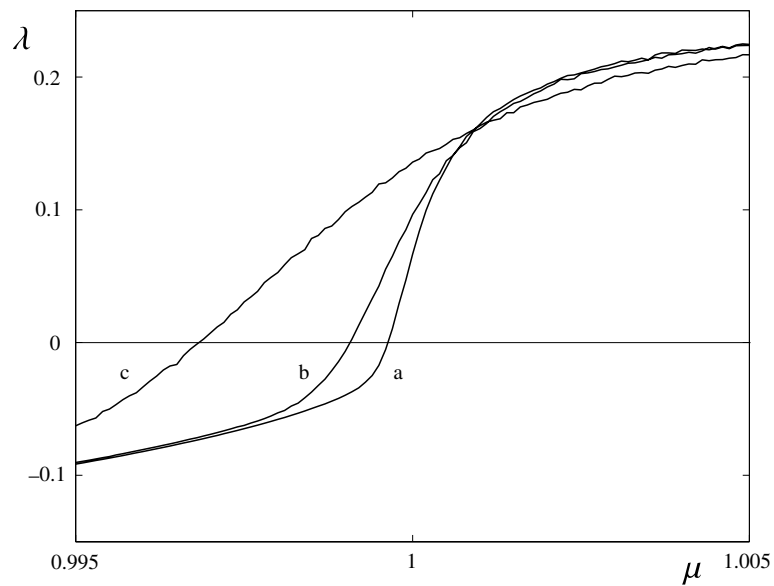


Fig. 7. Lyapunov exponents of stochastic system (2) for (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.002$, (c) $\varepsilon = 0.005$.

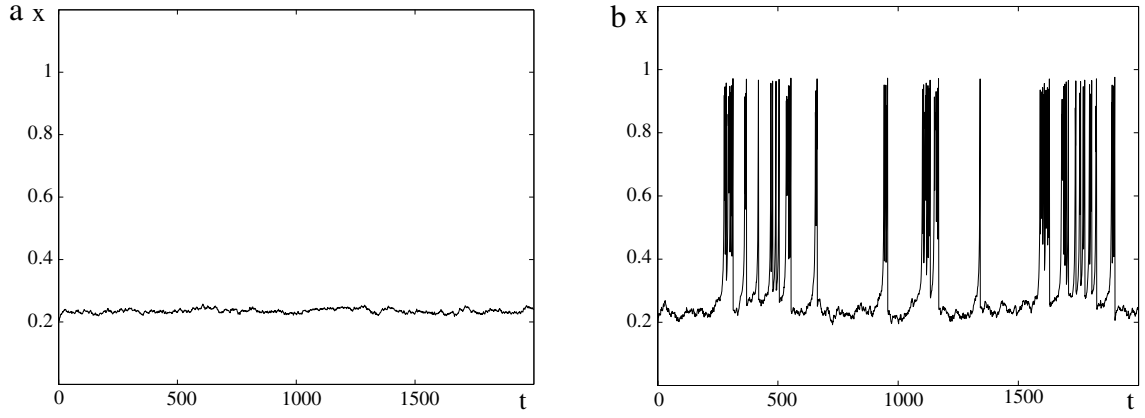


Fig. 8. Time series of stochastic system (2) with $\mu = 0.998$ for (a) $\varepsilon = 0.002$, (b) $\varepsilon = 0.005$.

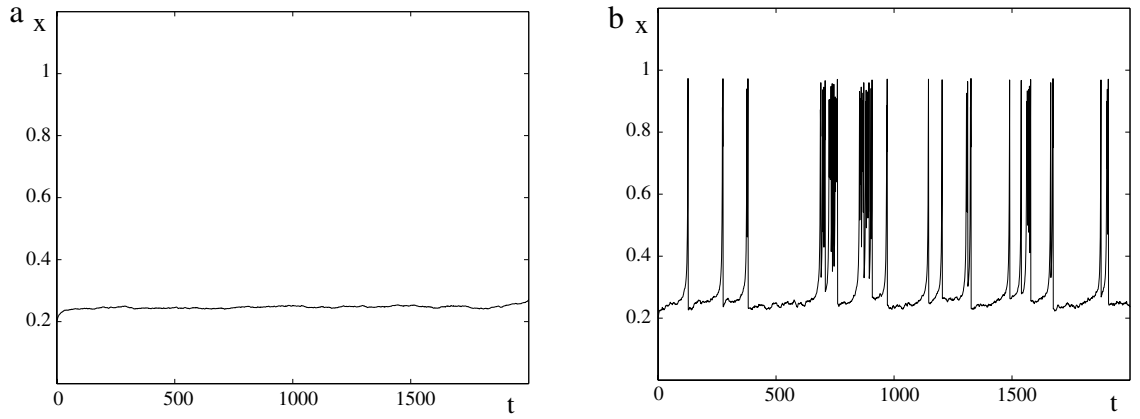


Fig. 9. Time series of stochastic system (2) with $\mu = 0.9998$ for (a) $\varepsilon = 0.0005$, (b) $\varepsilon = 0.002$.

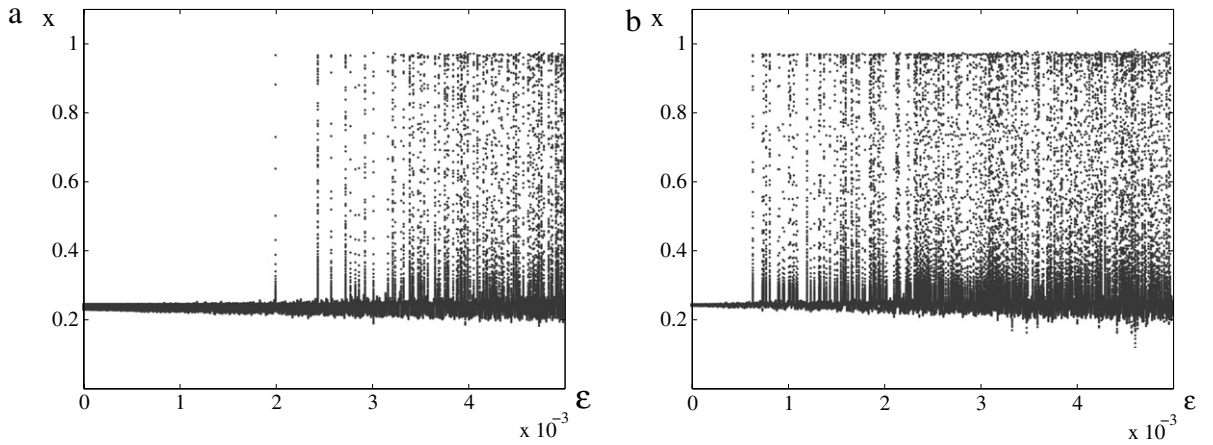


Fig. 10. Attractors of stochastic system (2) for (a) $\mu = 0.998$, (b) $\mu = 0.9998$.

3. Stochastic sensitivity analysis for noise-induced intermittency

3.1. SSF technique

Consider a nonlinear stochastic discrete-time system

$$x_{t+1} = f(x_t) + \varepsilon \sigma(x_t) \xi_t. \quad (3)$$

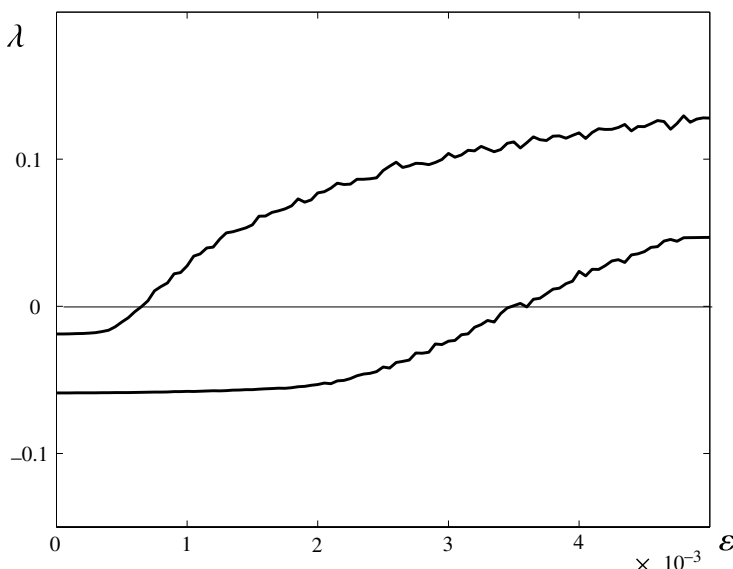


Fig. 11. Lyapunov exponents of stochastic system (2) for $\mu = 0.998$ (lower) and $\mu = 0.9998$ (upper).

Here the function $f(x)$ describes deterministic dynamics, and the function $\sigma(x)$ characterizes the dependence of random disturbances on state, ξ_t is uncorrelated Gaussian random process with parameters $E\xi_t = 0$, $E\xi_t^2 = 1$, ε is a noise intensity.

It is supposed that the system (3) for $\varepsilon = 0$ has an exponentially stable equilibrium $x_t \equiv \bar{x}$.

Let x_t^ε be a solution of the system (3) with the initial condition $x_0^\varepsilon = \bar{x} + \varepsilon v_0$. The variable

$$v_t = \lim_{\varepsilon \rightarrow 0} \frac{x_t^\varepsilon - \bar{x}}{\varepsilon}$$

characterizes the sensitivity of the equilibrium \bar{x} both to initial data disturbances and to the random disturbances of the system (3). For the sequence v_t , it holds that

$$v_{t+1} = \alpha v_t + \beta \xi_t, \quad \alpha = f'(\bar{x}), \quad \beta = \sigma(\bar{x}). \quad (4)$$

The dynamics of the moments $m_t = E v_t$, $w_t = E v_t^2$ for the solution v_t of the system (4) is governed by the equations

$$m_{t+1} = \alpha m_t, \quad w_{t+1} = \alpha^2 w_t + \beta^2. \quad (5)$$

For the exponentially stable equilibrium \bar{x} , it holds that $|\alpha| < 1$ and the solutions m_t and w_t of the system (5) are stabilized for any m_0 and w_0 :

$$\lim_{t \rightarrow \infty} m_t = 0, \quad w = \lim_{t \rightarrow \infty} w_t = \frac{\beta^2}{1 - \alpha^2}.$$

The mathematical details and proofs can be found in Ref. [26].

For small ε , a probabilistic distribution of x_t^ε is stabilized too. It means that the system (3) has a stationary distributed solution \bar{x}_t^ε with probability density function $p(x, \varepsilon)$. This function has the following Gaussian approximation:

$$p(x, \varepsilon) \approx \frac{1}{\varepsilon \sqrt{2\pi w}} \exp\left(-\frac{(x - \bar{x})^2}{2w\varepsilon^2}\right)$$

with mean value \bar{x} and dispersion $D = \varepsilon^2 w$. The value w connects the intensity of stochastic input (ε^2) with stochastic output (D) in the system (3) and characterizes a stochastic sensitivity of the equilibrium \bar{x} . For the stochastic sensitivity function w , the explicit formula can be written:

$$w = \frac{\sigma^2(\bar{x})}{1 - (f'(\bar{x}))^2}.$$

Values w and ε define the borders of the confidence interval (x_1^*, x_2^*) :

$$x_{1,2}^* = \bar{x} \pm k\varepsilon\sqrt{2w}.$$

Here the parameter k is connected with fiducial probability P by the formula $k = \text{erf}^{-1}(P)$, where $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ is the error function. It means that random states of the system (3) hit into this interval with the probability P . Confidence intervals characterize a spatial arrangement of random states of the system (3) near the stable equilibrium \bar{x} .

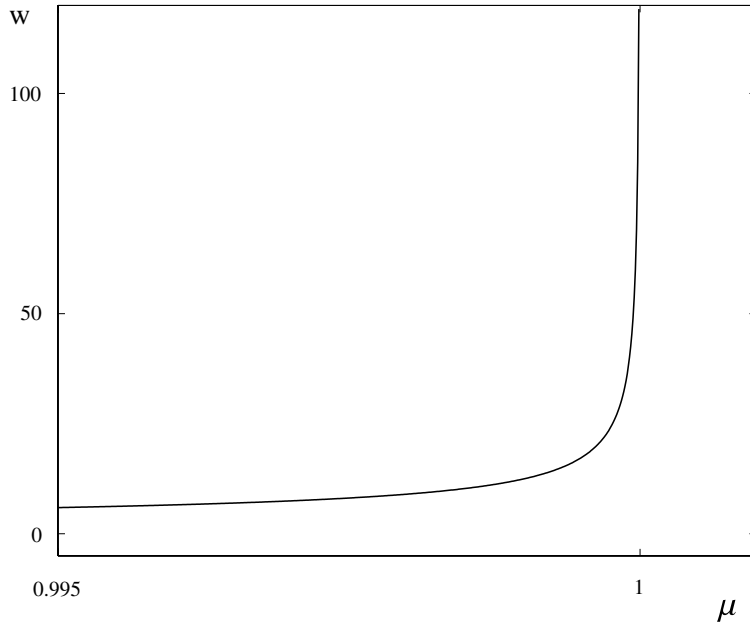


Fig. 12. Stochastic sensitivity function of the equilibrium $(\bar{x}_1(\mu))$.

The stochastic sensitivity function (SSF) technique was elaborated for the analysis of randomly forced equilibria and limit cycles for both continuous [25] and discrete-time [26] systems. The SSF technique was successfully applied to the study of some peculiarities of stochastic cycles in a period-doubling bifurcations zone [27].

Confidence domains are sufficiently simple and evident geometrical models for the spatial description of random states of the stochastic system. The SSF technique enables to construct confidence intervals, ellipses, bands and tori for the probabilistic analysis of various stochastic attractors [28,29].

In this paper, SSF technique and a method of confidence intervals are used for the constructive analysis of noise-induced intermittency.

3.2. Analysis of noise-induced intermittency

For quantitative analysis of the phenomenon of noise-induced intermittency in the system (2), we will use the corresponding stochastic sensitivity function.

For a zone $0.995 \leq \mu < 1$, the stochastic sensitivity function $w(\mu)$ of the stable equilibrium $\bar{x}_1(\mu)$ of the system (2) is the following:

$$w(\mu) = \frac{1}{1 - (f'(\bar{x}_1(\mu)))^2}.$$

Near the tangent bifurcation point, if $\mu \rightarrow \mu^* = 1$ then $f'(\bar{x}_1(\mu)) \rightarrow 1$ and $w(\mu) \rightarrow \infty$. In Fig. 12, the function $w(\mu)$ is plotted. Note that a high level of SSF values is the original cause of the noise-induced intermittency near the point of the tangent bifurcation.

Stochastic sensitivity function $w(\mu)$ allows us to construct confidence intervals $(x_1^*(\mu), x_2^*(\mu))$: $x_{1,2}^*(\mu) = \bar{x}_1(\mu) \pm k\varepsilon\sqrt{2w(\mu)}$. Here $k = \text{erf}^{-1}(P)$, P is a fiducial probability, ε is the noise intensity. These intervals characterize a dispersion of random states near the stable equilibrium $\bar{x}_1(\mu)$ and can be used in the quantitative analysis of phenomenon of noise-induced intermittency.

In Fig. 13, random states of the system (2) on the interval $0.995 \leq \mu < 1$ for three values of noise intensity are plotted in gray. Along with stochastic attractors, here one can see curves of the stable equilibrium $\bar{x}_1(\mu)$ (solid line), the unstable equilibrium $\bar{x}_2(\mu)$ (dashed line), and the borders $x_1^*(\mu), x_2^*(\mu)$ (dotted lines) of corresponding confidence intervals with fiducial probability $P = 0.99$. For the left part of the considered zone $0.995 < \mu < 1$, where $x_2^*(\mu) < \bar{x}_2(\mu)$, the results of the direct numerical simulation agree with the confidence intervals found by the SSF technique. Indeed, for this zone, the confidence intervals are entirely contained in the basin of the monotonous attraction of the stable equilibrium $\bar{x}_1(\mu)$ and therefore the random states of the system (2) are concentrated near $\bar{x}_1(\mu)$. At the right part of the interval $0.995 < \mu < 1$, because of the unlimited increase of the stochastic sensitivity and decrease of the distance $\bar{x}_2(\mu) - \bar{x}_1(\mu)$, the confidence intervals expand and for $x_2^*(\mu) > \bar{x}_2(\mu)$ begin to occupy the zone, where the system exhibits oscillations with large

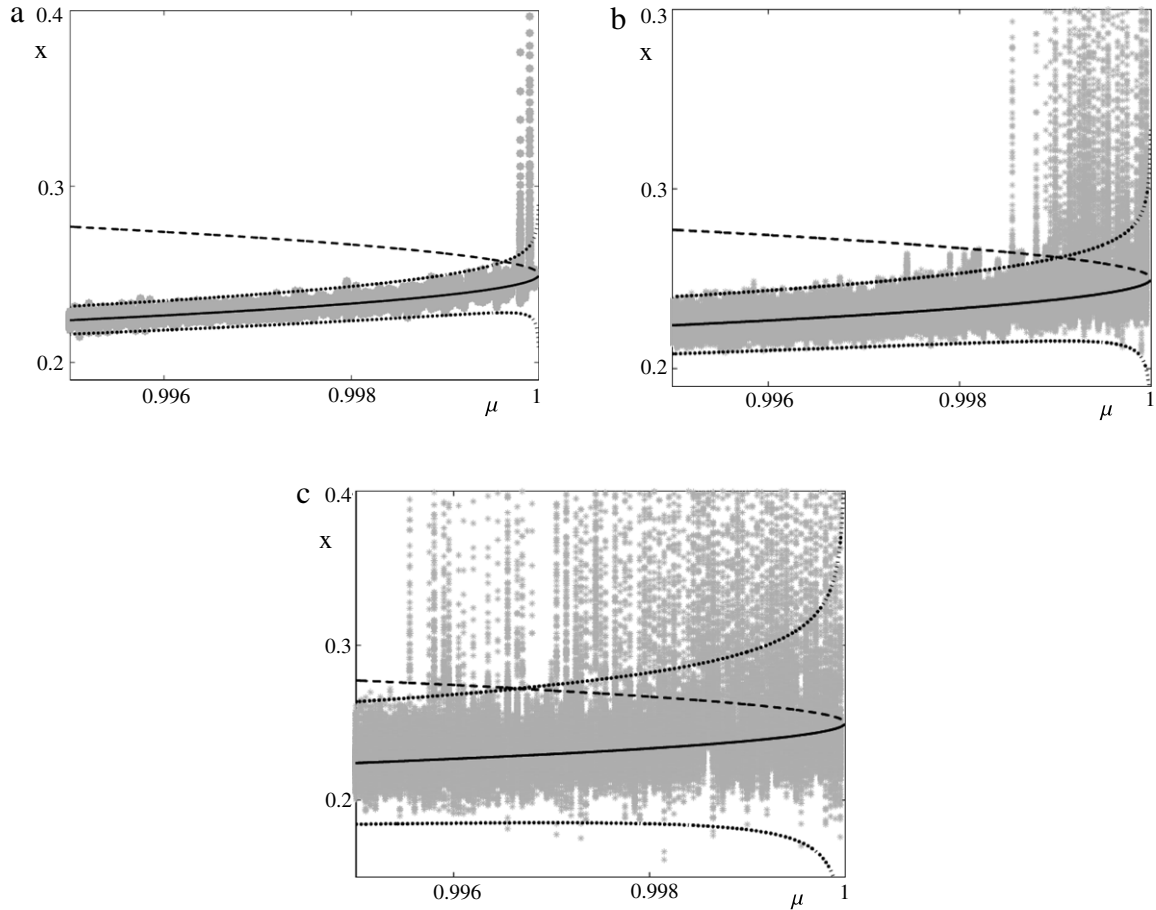


Fig. 13. Random states (gray), unstable equilibrium (dashed line), stable equilibrium (solid line) and borders of corresponding confidence intervals (dotted line) with $P = 0.99$ for (a) $\varepsilon = 0.001$, (b) $\varepsilon = 0.002$, (c) $\varepsilon = 0.005$.

amplitude. This occupation means that random trajectories of the noisy system with high probability can exceed the bounds of the unexcited regime and go on a large excursion.

Intersection point of the curves $x_2^*(\mu)$ and $\bar{x}_2(\mu)$ can be used as a marker for the border between unexcited regime with small amplitude stochastic oscillations and zone of excitement of chaotic oscillations. This point can be found from the equation

$$(\bar{x}_2(\mu) - \bar{x}_1(\mu))^2 = 2k^2\varepsilon^2w(\mu).$$

From this equation, an explicit formula for corresponding critical value ε^* of noise intensity follows:

$$\varepsilon^*(\mu) = \frac{|\bar{x}_2(\mu) - \bar{x}_1(\mu)|}{k\sqrt{2w(\mu)}}.$$

A plot of the function $\varepsilon^*(\mu)$ for the system (2) is shown in Fig. 14. This plot together with a line $\mu^* = 1$ distinguishes three zones. Here A is a zone of noisy order, B is a zone of noise-induced intermittency and C is a zone of noisy intermittency. Transition from A to B can be interpreted as noise-induced chaoticization. Thus, based on a SSF technique and a method of confidence intervals, one may construct parametrical portraits illustrating different regimes of stochastic dynamics for systems in which phenomenon of intermittency occurs.

3.3. Noise-induced intermittency and local asymptotics of tangent bifurcation

An essence of noise-induced intermittency can be described by local asymptotics of the function $f(x)$ in a zone of tangent bifurcation. For the asymptotic analysis, we consider the following one-dimensional model map:

$$x_{t+1} = f(x_t) + \varepsilon\xi_t, \quad f(x) = x - \delta(x - \bar{x}_1) + l|x - \bar{x}_1|^m + \text{"return"}'', \quad \delta \geq 0, \quad l > 0. \quad (6)$$

Here \bar{x}_1 is a fixed equilibrium, parameters δ , l and m define the asymptotic of $f(x)$ at the point of tangent bifurcation. The parameter δ characterizes a subcriticality, $m > 1$ is an order of contact of the curve $y = f(x)$ with $y = x$ at the point \bar{x}_1 for

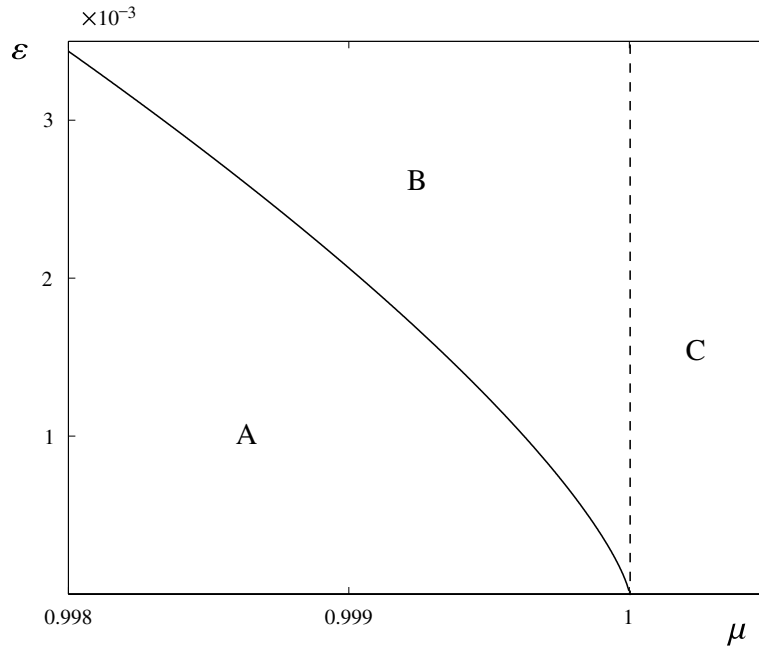


Fig. 14. Critical value $\varepsilon^*(\mu)$ (solid line) and zones: A—noisy order; B—noise-induced intermittency; C—noisy intermittency.

$\delta = 0$. For $\delta > 0$, the system (6) has an unstable equilibrium $\bar{x}_2 = \bar{x}_1 + (\delta/l)^{\frac{1}{m-1}}$ along with the stable equilibrium \bar{x}_1 . The return of the states of this system after large-amplitude oscillations to the vicinity of the equilibrium x_1 can proceed in different ways. Here the concrete mechanism of this return does not matter.

Using the SSF technique presented above, one can find the stochastic sensitivity function w and the confidence interval $(\bar{x}_1 - r, \bar{x}_1 + r)$:

$$w(\delta) = \frac{1}{2\delta - \delta^2}, \quad r = k\varepsilon \left(\delta - \frac{\delta^2}{2} \right)^{-\frac{1}{2}}.$$

Note that the stochastic sensitivity function depends on the parameter δ only.

An asymptotic of the critical value ε^* for noise intensity corresponding to the onset of noise-induced intermittency can be found from the equation $r(\varepsilon) = \bar{x}_2 - \bar{x}_1$ in the explicit form:

$$\varepsilon^* \approx \frac{1}{k} p \delta^q, \quad p = -\frac{1}{m-1}, \quad q = \frac{m+1}{2m-2}.$$

Note that the coefficient k depends on the fiducial probability P : $k = \text{erf}^{-1}(P)$.

4. Conclusion and discussion

We study an intermittency phenomenon for nonlinear systems forced by the random disturbances. Our paper is focused on the noise-induced intermittency and chaotization observed near tangent bifurcation. Through the study of a simple one-dimensional stochastic system, we present the main probabilistic phenomena and methods of their analysis. The remarkable feature of the dynamics of the model considered here is that small noises generate large-amplitude chaotic oscillations even in the subcritical zone where the deterministic system has a single stable equilibrium. We show that this phenomenon can be explained by the high stochastic sensitivity of this equilibrium. In this paper, for the probabilistic distribution of random states in the stochastic attractor, we use the constructive approximation based on the stochastic sensitivity function technique. On the basis of the SSF technique, we find confidence intervals for these random states and use them in the parametrical analysis of noise-induced intermittency. For a sufficiently small noise, the confidence intervals are localized near the stable equilibrium. As the noise intensity increases, the confidence intervals expand and begin to occupy the zone of large-amplitude oscillations. This occupation means that random trajectories of the forced system with a high probability can exceed the bounds of the unexcited regime and go on a large excursion generating chaotic oscillations. The noise intensity that corresponds to the beginning of this occupation can be used as the estimation of the threshold value. In the present work, we have found an explicit formula for the value of noise intensity threshold corresponding to the onset of noise-induced intermittency and constructed a parametrical diagram of different stochastic regimes. Our method enables to determine the asymptotic of the critical noise intensity as a function of parameters of tangent bifurcation in a general case.

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